J. Fluid Mech. (1999), *vol.* 399, *pp.* 301–318. Printed in the United Kingdom

The production of subharmonic waves in the nonlinear evolution of wavepackets in boundary layers

By MARCELLO A. F. MEDEIROS† AND MICHAEL GASTER‡

Cambridge University Engineering Department, Trumpington Street, Cambridge CB2 1PZ, UK

(Received 9 March 1998 and in revised form 1 July 1999)

The nonlinear evolution of wavepackets in a laminar boundary layer has been studied experimentally. The packets were generated by acoustic excitations injected into the boundary layer through a small hole in the plate. Various packets with different phases relative to the envelope were studied. It was found that for all the packets the nonlinearity involved the appearance of oblique modes of frequency close to the subharmonic of the dominant two-dimensional wave. Moreover, the results confirmed that the phase had a strong influence on the strength of the nonlinear interaction. The experimental observations also indicated that although a subharmonic resonance appeared to be present in the process, it alone could not explain the nonlinear behaviour. The experiment demonstrated that the process must also involve a mechanism that generates oblique waves of frequency lower than the Tollmien–Schlichting band.

1. Introduction

Laminar–turbulent transition is an active area of research in fluid dynamics. In particular, the evolution of wavepackets in a laminar flat-plate boundary layer have been studied. This type of oscillation has been shown to evolve in a manner similar to that of naturally occurring waves (Gaster 1978*a*; Shaikh 1993), and therefore is of significant practical interest. Often the wavepackets studied are generated from a pulse excitation, but other types can also be generated. In Medeiros & Gaster (1999) the evolution of different types of wavepacket was studied experimentally. There the problem was introduced by the comparison of the evolution of two wavepackets formed respectively from a positive and a negative short-duration acoustic pulse. The experimental observations showed that the nonlinear behaviour of the packets is strongly influenced by the sign of the excitation. It had been suggested previously (Y. S. Kachanov 1994 and A. Seifert 1995, personal communications) that this somewhat surprising behaviour arose because disturbances of opposite sign produce different nonlinear effects close to the excitation source and that this would explain the observed downstream behaviour. However, the experimental results discussed in Medeiros & Gaster (1999) have shown that this is not the case. A natural extension

† Present Address: Departamento de Engenharia Mecânica, Pontifícia Universidade Católica de Minas Gerais, Rua Dom José Gaspar, 500, Belo Horizonte, 30535-610, MG, Brazil.

‡ Present Address: Department of Mechanical Engineering, Queen Mary & Westfield College, Mile End Road, London, UK.

to the simple positive and negative pulse excitations can be found by treating the amplitude of the pulse as a complex quantity. This form of excitation includes the positive and negative pulse, but also enables one to consider the more generic forms. As was shown in Medeiros & Gaster (1999) the wavepackets created by these generalized pulses contain wavetrains of different phases embedded within the same envelope when the amplitude magnitudes are small, but which are quite different in the nonlinear regime. These results suggested that the phenomenon was linked to the phase of the packet relative to the envelope, but no explanation for this interesting behaviour has so far been given.

In other experiments with wavepackets (Gaster & Grant 1975; Cohen, Breuer & Haritonidis 1991), as well as in some numerical simulations (Konzelmann 1990) the appearance of oblique modes marked the onset of the nonlinear regime. In some of these studies the phenomenon was attributed to some type of subharmonic resonance. In those studies, however, only packets generated from a positive pulse were considered. The subharmonic resonance is sensitive to the relative phase of the resonant modes (Monkewitz 1988), and thus perhaps could offer an explanation as to why the phase of the ripples affects the evolution of a packet. In Medeiros & Gaster (1999) the measurements that were discussed were restricted to the centreline of the flow, and it seemed that more comprehensive experiments were necessary to shed further light on the problem. Here measurements were taken at a number of spanwise positions in order to compose a three-dimensional view of the wave structure. To provide an extension to the previous work, the new experiments were carried out at the same experimental conditions.

The results confirmed the dominant influence of the phase of the ripples within the packet envelope on the evolution of the packet downstream. The appearance of oblique modes was a common feature of the different packets when nonlinearity became evident. These modes had frequencies close to the subharmonic of the Tollmien–Schlichting waves within the packet. New experiments to be discussed here were specially designed to provide suitable data for showing whether or not there was a subharmonic resonance. The deterministic resonant mechanism can only be triggered by an appropriate low-frequency seed. These experiments show that, if a resonant mechanism is present, the seed is not directly connected to the controlled excitation.

2. Experimental observations

The experimental set up and procedures were the same as those used in Medeiros & Gaster (1999), with the exception that here a number of spanwise measuring stations were covered. The experiment was totally controlled by the computer and took some 80 hours of continuous operation of the tunnel. In such a long experiment the variation of environmental conditions may cause serious problems because the viscosity of the air is very sensitive to both temperature and pressure. For instance, a variation of 1 °C in the temperature corresponds to variations of order 1% in the Reynolds number, which in some circumstances are unacceptable. In the current series of experiments great care was taken to ensure that this variation was kept to a minimum by controlling the laboratory temperature. Despite all efforts, the variation of R was still about $\pm 2\%$ (Medeiros 1996). Fortunately this did not seem to have any substantial influence on the data.

An overall view of the development of the wavepackets is given in figure 1. Each contour plot was constructed from hot-wire records taken at 41 different spanwise stations, separated from each other by 1 cm. The measurements thus covered the entire

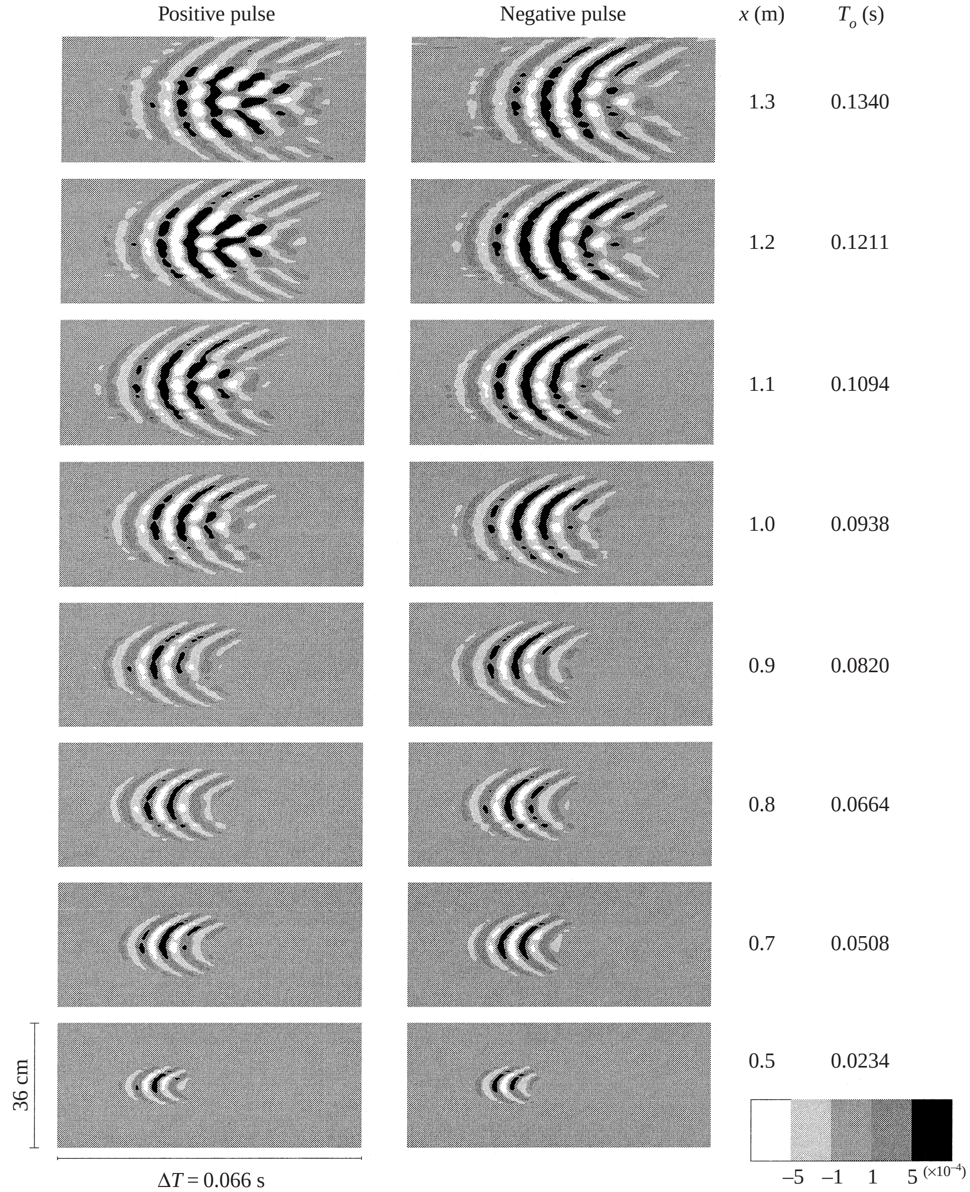

FIGURE 1. Comparison of the evolution of the positive and the negative wavepackets on a time-span plane shown by contour plots of the streamwise velocity fluctuation non-dimensionalized by the free-stream velocity. The measurements were taken at $y = 0.6\delta^*$. T_o is the time delay between the excitation and the beginning of the interval ΔT.

width of the packet. The observations indicated that the nonlinearity was substantially stronger inside the boundary layer than outside. The records shown here were taken at $y/\delta^* = 0.6$, which is close to the peak of the Tollmien–Schlichting eigenfunction. The view represents a time history of the velocity fluctuations as the packet passed through a fixed streamwise location and must not be taken as a spatial picture of the flow at some time instant. The wave fronts are in the form of crescents which increase in spanwise extent as the disturbance progresses downstream.

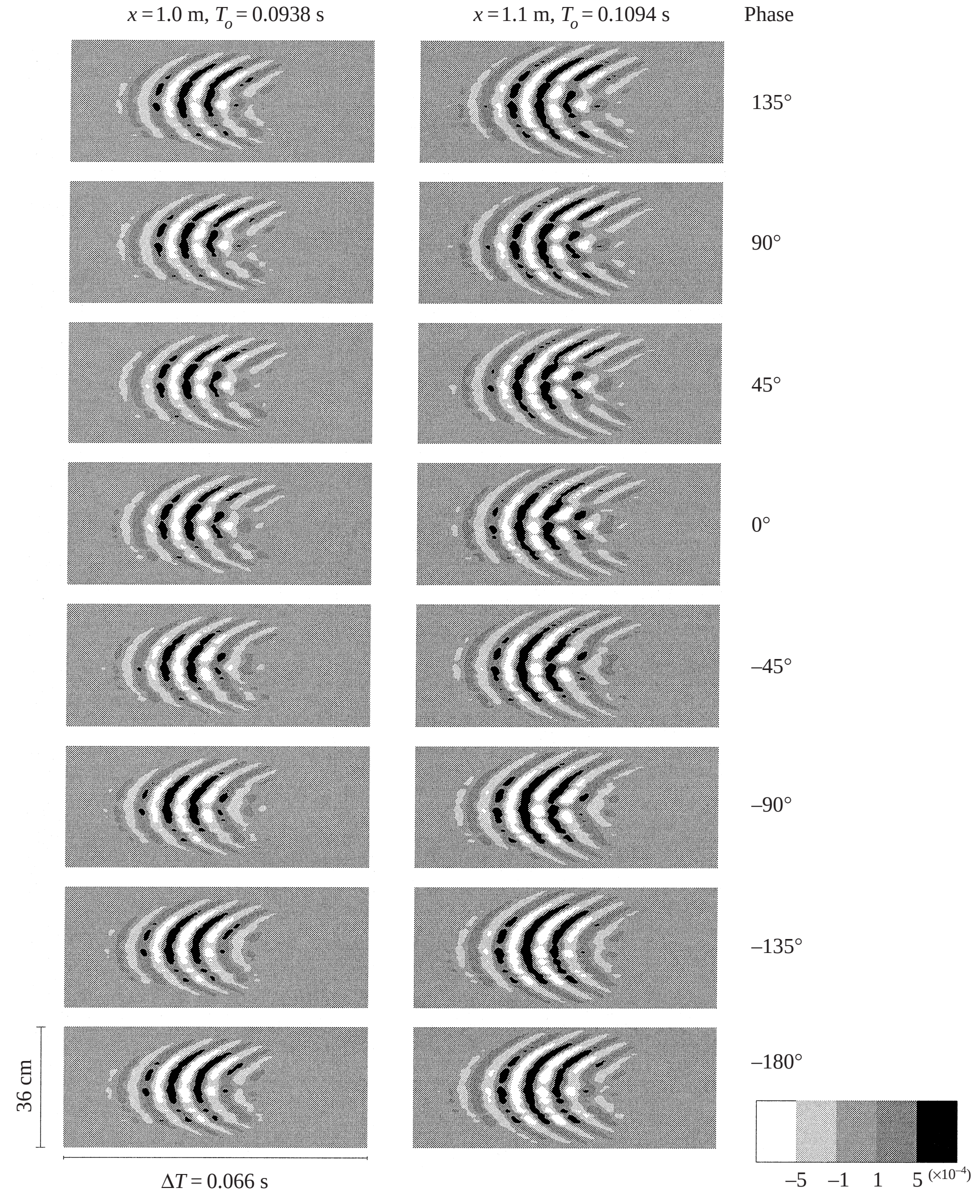

FIGURE 2. For caption see facing page.

Initially the wave crests of the packets are smooth, both for the packet originated from a positive pulse (here called positive packet) and the packet from the negative pulse (the negative packet). As the nonlinearity develops, distortions arise and the wave fronts become warped, but even at the last streamwise measuring station the crests of the negative packet are still comparatively smooth. The three-dimensional structure that is observed in the nonlinear regime is similar for packets of different phases relative to the envelope, figure 2, but clearly the strength of the nonlinear interaction was dependent on the phase of the packet.

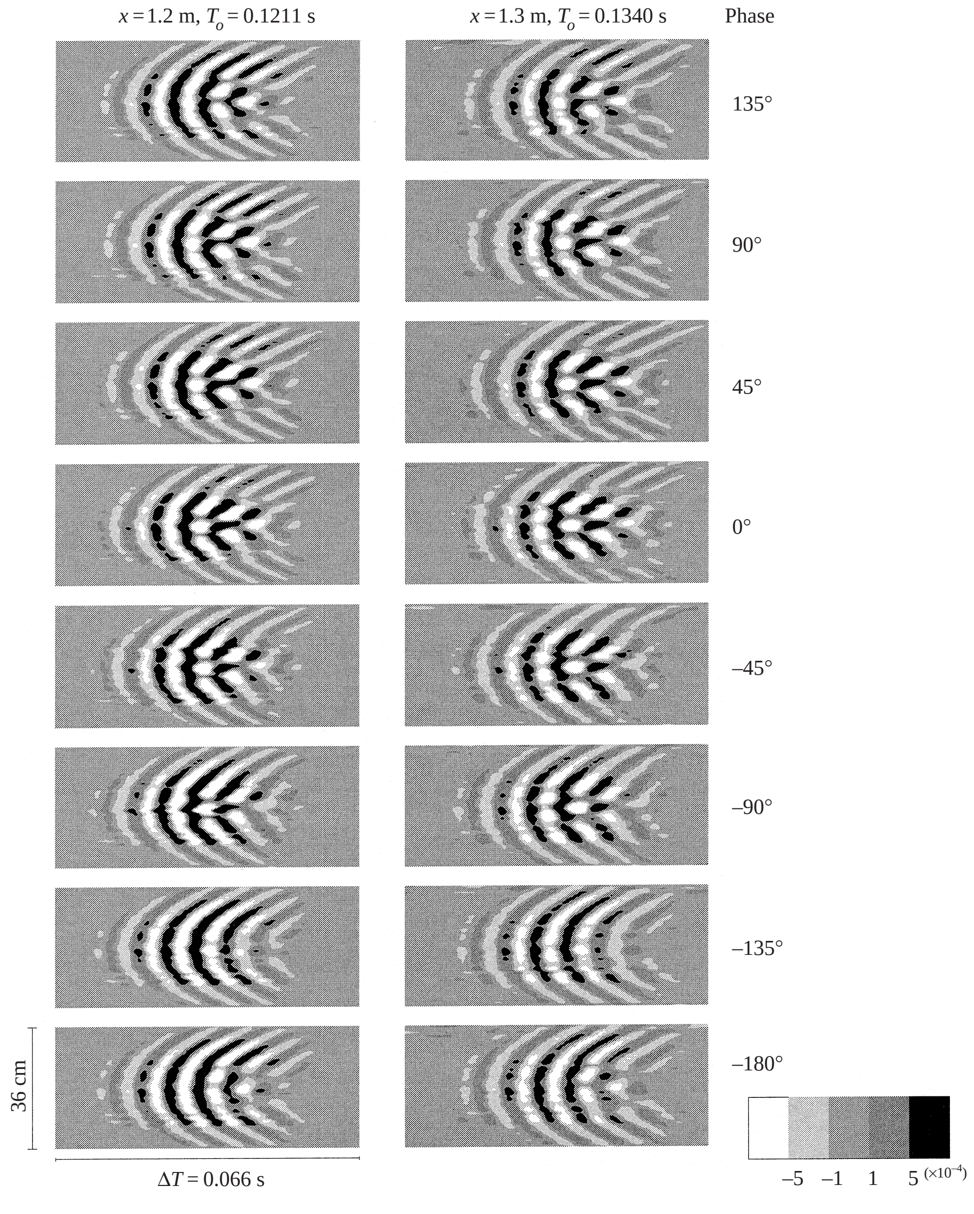

FIGURE 2. Variation of the nonlinear behaviour with the wavepacket phase on a time-span plane shown by contour plots of the streamwise velocity fluctuation non-dimensionalized by the free-stream velocity. The measurements were taken at $y = 0.6\delta^*$.

A more definitive view concerning the nature of the warped crests observed in the nonlinear regime of the disturbance can be obtained in the Fourier domain. Figure 3 displays power spectra of the signals in the non-dimensional frequency(β) × spanwise wavenumber(α_z) plane. Initially the energy is concentrated at $\beta = 0.1$, which corresponds to the linearly most amplified waves. In the linear regime, the wave crests of the packets are smooth and the spanwise wavenumbers concentrate around zero. In the late stages of the development of the positive packet there is a sharp increase in

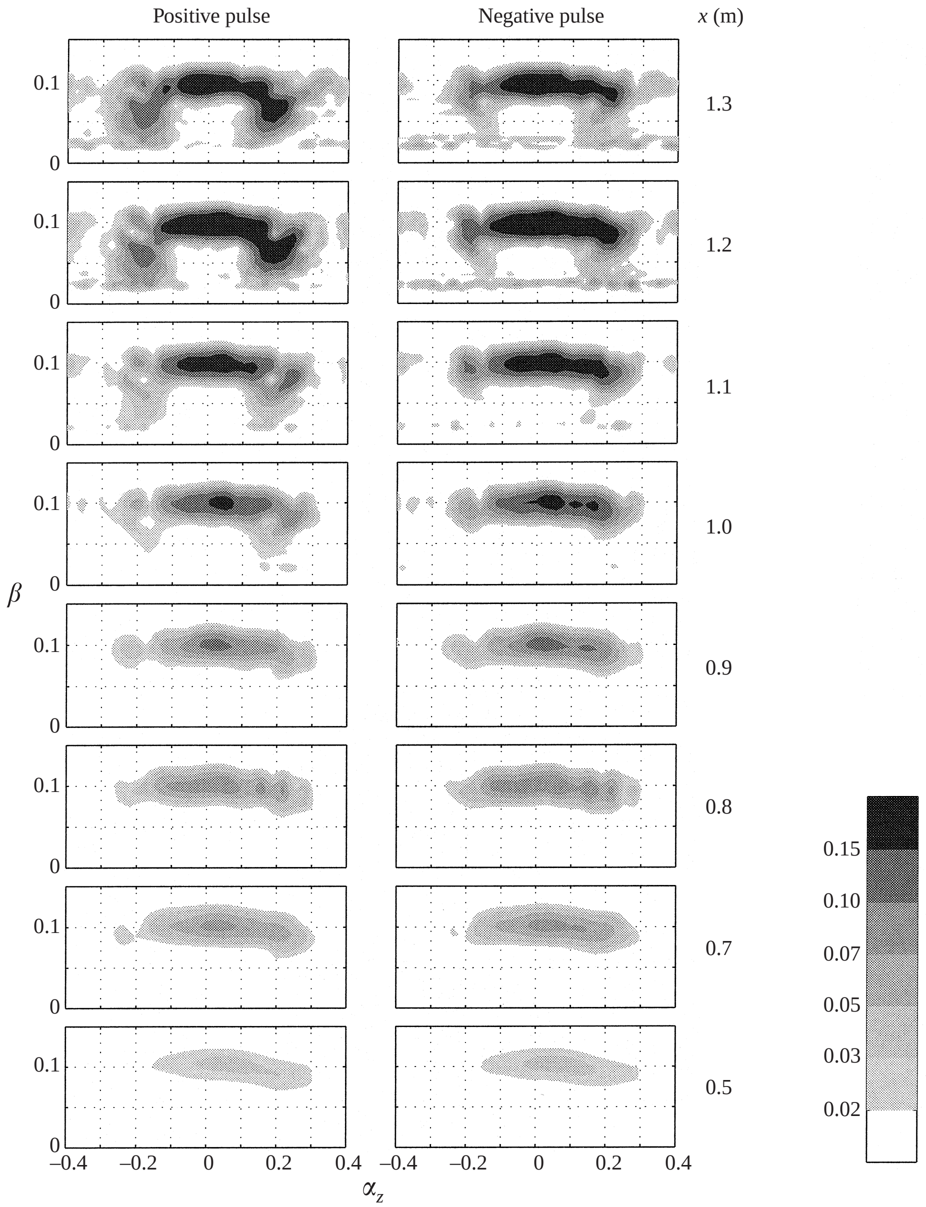

FIGURE 3. Comparison of the evolution of the positive and the negative wavepackets in the frequency ($\beta = 2\pi f\delta^*/U_\infty$) × spanwise wavenumber ($\alpha_z = 2\pi\delta^*/\lambda_z$) plane, where f is the dimensional frequency and λ_z is the dimensional spanwise wavelength. The plots show the relative magnitude. Measurements taken at $y = 0.6\delta^*$.

the energy of modes with frequencies lower than the Tollmien–Schlichting band and non-zero spanwise wavenumber. As occurs with nonlinear plane regular wavetrains, the nonlinear regime of the wavepacket is associated with the appearance of oblique waves. In the Fourier domain the difference in the evolution of the positive and the negative wavepackets is more clearly seen. The dependence of the results on the phase of the packet is shown in figure 4. It is observed that the spanwise wavenumber of the oblique waves is roughly the same for all the nonlinear packets. For the experimental conditions used, the phase $-135°$ appeared to be the most resistant to the nonlinear mechanism. Indeed for this phase very little energy is detected outside the Tollmien–Schlichting band even at the last streamwise station.

3. Some theoretical considerations

The first impression that arises from figure 3 is that the nonlinear interaction is associated with the appearance of subharmonic oblique waves, which has also been observed in nonlinear regular plane wavetrains and led to the development of a number of theories. Among these theories two were particularly successful, namely, the three-wave resonance proposed by Craik (1971) and the secondary instability developed by Herbert (1988). They do not take into account the modulation of the wave, but show good agreement with experiments on plane regular wavetrains (Corke & Mangano 1989). A brief review of them is presented here; details can be found in the quoted references.

The standard procedure used to analyse the linear stability of flows involves the decomposition of the velocity field and the pressure field into a base component $\boldsymbol{V}, P$, which is a solution of the steady equations of motion, and a small disturbance part $\boldsymbol{v}, p$. Substituting into the Navier–Stokes equations, subtracting out the base flow and neglecting the quadratic terms, one arrives at a linear system of equations describing the disturbance field:

$$\frac{\partial}{\partial t}\boldsymbol{v} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{V} = -\nabla p + \frac{1}{R}\nabla^2\boldsymbol{v}, \tag{1}$$

$$\nabla\cdot\boldsymbol{v} = 0. \tag{2}$$

The coefficients of these equations are given by the base flow solution. For the boundary layer the equations of motion are non-dimensionalized by the free-stream velocity U_∞ and the displacement thickness δ^*; therefore $R = U_\infty\delta^*/\nu$, where ν is the kinematic viscosity. With the additional assumption that the base flow is parallel, that is, $\boldsymbol{V} = (U, 0, 0)$, the system of equations (1) and (2) permits the normal modes solution

$$\begin{bmatrix} u \\ v \\ w \\ p \end{bmatrix} = \begin{bmatrix} \hat{u}(y) \\ \hat{v}(y) \\ \hat{w}(y) \\ \hat{p}(y) \end{bmatrix} \mathrm{e}^{\mathrm{i}(\alpha_x x + \alpha_z z - \beta t)}, \tag{3}$$

where u, v, w represent the velocity components in the streamwise, normal to the wall and spanwise directions (x, y, z), respectively, and p is the pressure; β is the non-dimensional frequency, and α_x and α_z are, respectively, the non-dimensional streamwise and spanwise wavenumbers. Both α_x and β are in general complex and account for the growth or decay of the waves in space and time (Gaster 1962, 1965). The functions $\hat{u}(y), \hat{v}(y), \hat{w}(y)$ and $\hat{p}(y)$ are also complex and define the structure of the mode through the boundary layer.

Substituting (3) together with $\boldsymbol{V} = (U, 0, 0)$ in (1) and (2), the equations of motion for a three-dimensional disturbance reduce to a pair of ordinary differential equations (Squire 1933; Mack 1984; Cohen *et al.* 1991)

$$\left\{\frac{\mathrm{d}^4}{\mathrm{d}y^4} - 2k^2\frac{\mathrm{d}^2}{\mathrm{d}y^2} + k^4 - \mathrm{i}R\alpha_x\left[(U-c)\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - k^2\right) - \frac{\mathrm{d}^2U}{\mathrm{d}y^2}\right]\right\}\hat{v} = 0, \tag{4}$$

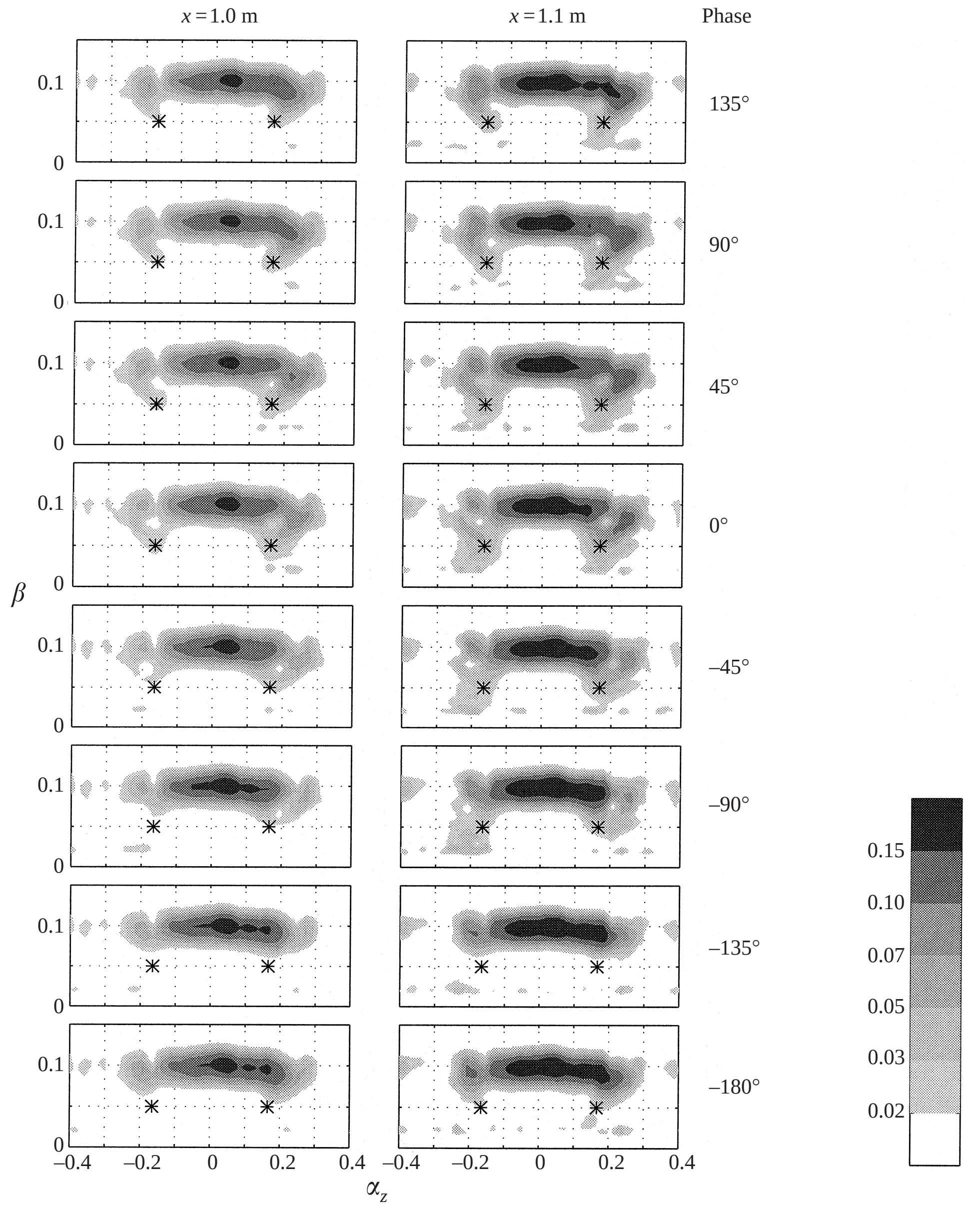

FIGURE 4. For caption see facing page.

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} - k^2 - \mathrm{i}R\alpha_x(U-c)\right]\hat{\eta} = \mathrm{i}\alpha_z R\frac{\mathrm{d}U}{\mathrm{d}y}\hat{v}, \tag{5}$$

with boundary conditions

$$\left.\begin{array}{lll} \text{at } y=0: & \hat{v}(y)=0, \quad \dfrac{\partial}{\partial y}\hat{v}(y)=0, \quad \hat{\eta}(y)=0, \\ \text{at } y\to\infty: & \hat{v}(y)\to 0, \quad \dfrac{\partial}{\partial y}\hat{v}(y)\to 0, \quad \hat{\eta}(y)\to 0, \end{array}\right\} \tag{6}$$

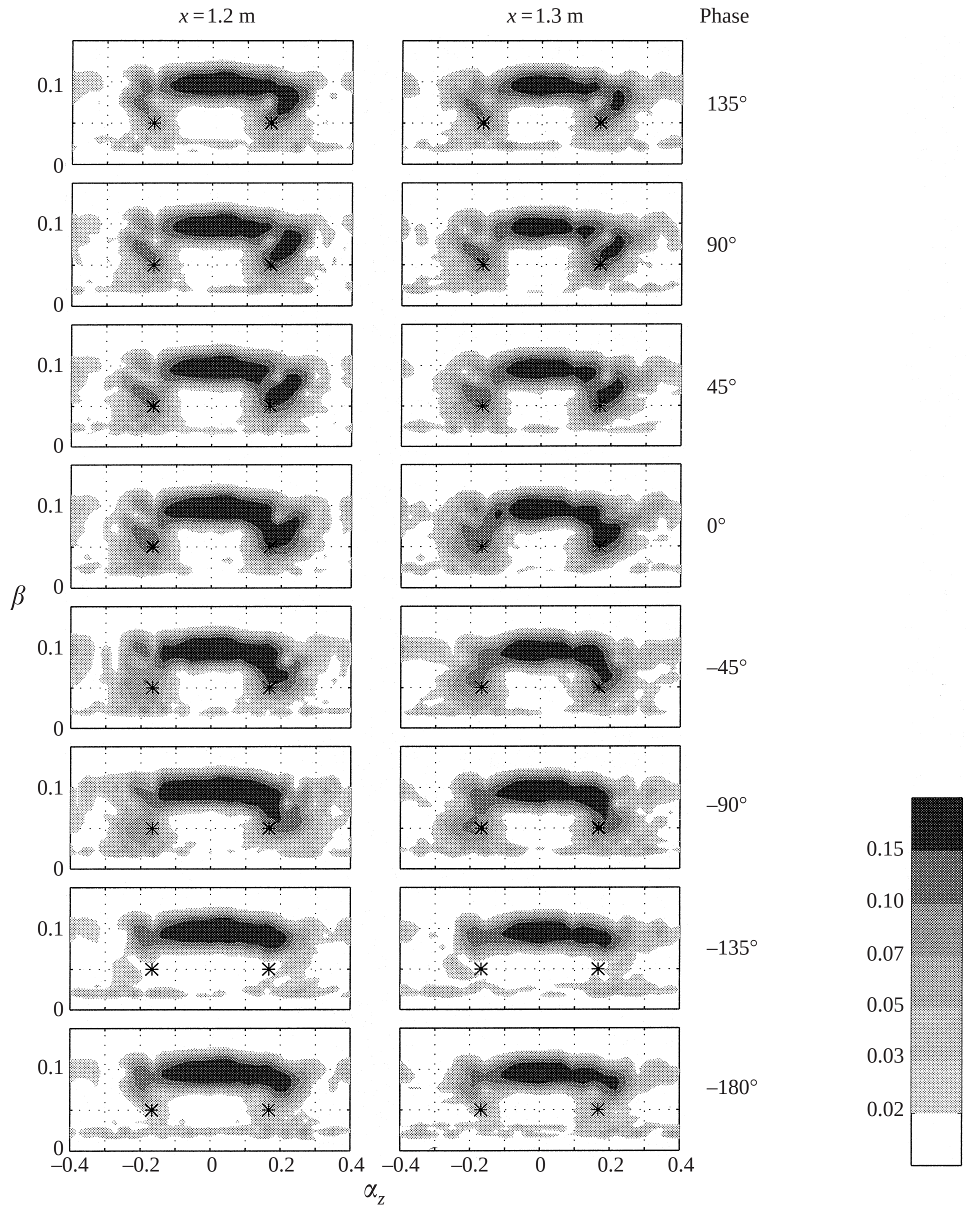

FIGURE 4. Variation of the nonlinear behaviour with the wavepacket phase shown in the frequency $(\beta) \times$ spanwise wavenumber (α_z) plane. Measurements taken at $y = 0.6\delta^*$. The plots show the relative magnitude. The asterisks indicate the waves satisfying Craik's resonance condition.

where $k^2 = \alpha_x^2 + \alpha_z^2$, c $(= \beta/\alpha_x)$ is the phase velocity of the mode and η $(= \partial w/\partial x - \partial u/\partial z)$ is the vertical vorticity.

Equation (4) is the Orr–Sommerfeld equation which, together with the associated homogeneous boundary conditions, constitutes an eigenvalue problem. Non-trivial solutions, or modes, have to satisfy a dispersion relation

$$F(\alpha_x, \alpha_z, \beta, R) = 0 \tag{7}$$

and each mode is associated with an eigenfunction $\hat{v}$.

The eigenfunctions for $\hat{u}(y)$ and $\hat{w}(y)$ require the solution of equation (5) for the vertical vorticity together with

$$\hat{u} = \frac{\mathrm{i}}{k^2}\left(\alpha_x \frac{\mathrm{d}\hat{v}}{\mathrm{d}y} - \alpha_z \hat{\eta}\right), \tag{8}$$

and

$$\hat{w} = \frac{\mathrm{i}}{k^2}\left(\alpha_z \frac{\mathrm{d}\hat{v}}{\mathrm{d}y} - \alpha_x \hat{\eta}\right). \tag{9}$$

These equations have successfully described the evolution of Tollmien–Schlichting waves of small amplitude in the boundary layer, including the wavepacket, which was modelled by Gaster (1975) as a superposition of normal modes. When the wave amplitudes are large the linear model fails and a number of nonlinear theories have been proposed to extend the model.

If one takes into account the background flow noise, the wave system will inevitably involve additional bands of waves to those artificially introduced, even when a monochromatic wave is excited in the boundary layer. Craik developed a theory to model the interaction of waves when a number of them are present in the flow. These wave interactions arise from the nonlinear terms of the equations of motion that were neglected in the linear approximation, but which become increasingly significant as the wave amplitude increases. A particularly strong interaction occurs if the waves resonate. Following Craik (1985) suppose that three dominant linear waves have the form

$$a_j(t)\mathrm{e}^{\mathrm{i}(\alpha_{xj}x + \alpha_{zj}z - \beta_j t)}, \quad j = 1, 2, 3 \tag{10}$$

with a_j small. In his analysis Craik considers temporal modes, therefore α_x is real. In the expression β is also real and the nonlinear growth or decay of the wave is given by $a(t)$. Owing to the quadratic nature of the nonlinear terms, the interaction of these modes yields $O(a^2)$ terms which have the form

$$\mathrm{e}^{\mathrm{i}[\pm(\alpha_{xp}x + \alpha_{zp}z - \beta_p t) \pm (\alpha_{xq}x + \alpha_{zq}z - \beta_q t)]}, \tag{11}$$

for two modes with $j = p$ and $j = q$.

The wave resonance occurs when any of these quadratic terms has the same periodicity as that of the third wave mode, that is, when

$$\alpha_{x1} \pm \alpha_{x2} \pm \alpha_{x3} = 0, \tag{12}$$

$$\alpha_{z1} \pm \alpha_{z2} \pm \alpha_{z3} = 0, \tag{13}$$

$$\beta_1 \pm \beta_2 \pm \beta_3 = 0, \tag{14}$$

with corresponding signs being chosen. Craik studied an interaction involving one two-dimensional wave $(\alpha_{x3}, 0, \beta_3)$ and two oblique waves $(\alpha_{x1}, \alpha_{z1}, \beta_1)$ and $(\alpha_{x2}, -\alpha_{z1}, \beta_2)$ travelling at opposite angles. To satisfy the resonance condition

$$\alpha_{x1} = \alpha_{x2} = \tfrac{1}{2}\alpha_{x3} \tag{15}$$

and

$$\beta_1 = \beta_2 = \tfrac{1}{2}\beta_3. \tag{16}$$

Craik suggested that through this resonant interaction a two-dimensional wave could selectively amplify modes from the background noise and impart a preferred spanwise periodicity to the flow. The possibility of resonant interaction can be verified from the linear solution of the problem. Accordingly, for a given two-dimensional wave, α_{x1} and α_{x2} are taken as $\frac{1}{2}\alpha_{x3}$. Using the dispersion relation one searches for oblique waves such that $\beta_1 = \beta_2 = \frac{1}{2}\beta_3$. The procedure determines α_{z1}.

Craik obtained a set of equations describing the evolution of the amplitude of the waves under the influence of such a resonant mechanism:

$$\frac{\mathrm{d}a_1}{\mathrm{d}t} + \sigma_1 a_1 = \lambda_1 a_2^* a_3 + O(a^3), \tag{17}$$

$$\frac{\mathrm{d}a_2}{\mathrm{d}t} + \sigma_2 a_2 = \lambda_2 a_1^* a_3 + O(a^3), \tag{18}$$

$$\frac{\mathrm{d}a_3}{\mathrm{d}t} + \sigma_3 a_3 = \lambda_3 a_1 a_2 + O(a^3), \tag{19}$$

where the λ are evaluated from the linear solution (u, v, w). The asterisk denotes complex conjugates. It is readily seen that for small amplitudes the equations (17) to (19) become independent of one another and the solution is the exponential growth predicted by linear theory. If the nonlinear terms are considered, the growth rates are modified and become dependent on the amplitude of the other interacting modes. Experimental confirmation of the phenomenon in the boundary layer was reported by Corke & Mangano (1989). They artificially excited the flow with three waves satisfying the resonance criterion and observed an enhancement of the growth rates of the waves.

A less restrictive resonant mechanism was proposed by Herbert (1983, 1988). He attributed the failure of the system (4) to (6) to the fact that with finite Tollmien–Schlichting waves the base flow is no longer steady. It is an almost periodic flow composed of two parts

$$\boldsymbol{V}(x, y, t) = \boldsymbol{V}_L(y) + A\boldsymbol{V}_{TS}(x, y, t), \tag{20}$$

where $\boldsymbol{V}_L$ represents the boundary layer flow and $\boldsymbol{V}_{TS}$ the Tollmien–Schlichting wave with amplitude A. Considering a frame of reference moving with the Tollmien–Schlichting phase velocity c_r, the basic flow is independent of time and satisfies

$$\boldsymbol{V}(x', y) = \boldsymbol{V}(x' + \lambda_x, y), \quad x' = x - c_r t, \tag{21}$$

where λ_x is the wavelength of the streamwise periodicity of the base flow.

The linearized disturbance equation (1) with $\boldsymbol{V}$ given by (20) constitutes a system with x-periodic coefficients. If one considers a locally parallel flow, which also constrains A to being locally constant, the normal mode concept can still be applied to z and t. Therefore, three-dimensional disturbances are written as

$$\boldsymbol{v} = (x', y, z, t) = \mathrm{e}^{\mathrm{i}(\alpha_z z - \beta t)}\hat{\boldsymbol{v}}(x, y). \tag{22}$$

The resulting equation and the boundary conditions of the problem are homogeneous. The system permits various classes of solutions, such as the primary (fundamental) resonance type and the principal parametric (subharmonic) resonance type, which are analysed through Floquet theory (Drazin & Reid 1981; Grimshaw 1990). The waves that arise from fundamental resonance have the same periodicity of the base flow (λ_x). Those arising from subharmonic resonance have wavelength $2\lambda_x$. Whereas Craik's mechanism only permits amplification of waves of a particular spanwise wavenumber, for the secondary instability the range of amplified spanwise wavenumber broadens as the amplitude of the Tollmien–Schlichting wave increases.

However, for a wavepacket composed of many two- and three-dimensional waves, Craik's resonance mechanism given by equations (17)–(19) can also explain the broad spectrum of the low-frequency oblique modes, provided it is not restricted to a single pure two-dimensional fundamental mode. Moreover, when the secondary instability theory is applied to a wavepacket, the physical meaning of the constant amplitude A of the two-dimensional Tollmien–Schlichting wave, in equation (20), is not obvious.

Craik's and Herbert's approaches to the nonlinear problem are conceptually different and lead to quite different formulations of the problem. However, for Tollmien–Schlichting waves of small enough amplitudes the parametric resonance should reproduce Craik's criterion for three-wave resonance. In fact, from the Floquet theory it was found that for small amplitudes the nonlinear regime is dominated by Craik's resonance (Herbert 1988). As the amplitudes increase, secondary instability sets in.

4. Analysis of the results

After Gaster successfully modelled the linear wavepacket, a few other works investigated the nonlinear regime of these waves in the boundary layer. The papers by Cohen *et al.* (1991), Cohen (1994), Breuer, Cohen & Haritonidis (1997) and Kozelmann & Fasel (1991) are important examples. In these the appearance of oblique waves was attributed to some sort of resonance mechanism. In all cases the two-dimensional Tollmien–Schlichting wave driving the resonance interaction was considered to be the dominant two-dimensional mode of the wavepacket. Using the Orr–Sommerfeld equation Cohen *et al.* calculated the oblique modes satisfying Craik's criterion for resonance and found good agreement with their experimental results. Konzelmann & Fasel interpreted the appearance of oblique waves as secondary instability of the subharmonic resonance type. The apparent disagreement between these studies might be explained by the amplitudes of the Tollmien–Schlichting waves used in the two cases. In the experiments by Cohen *et al.* the fundamental waves were very small which probably favoured Craik's mechanism. It is possible that Konzelmann & Fasel used larger excitation amplitudes in their simulation which would have favoured secondary instability. There is indeed some indication that the amplitudes in the simulations were relatively large.

Regarding the observation that the nonlinear interaction is stronger inside the boundary layer, Cohen *et al.* offered a plausible explanation. From equation (5) it is observed that, for oblique waves, the eigenfunctions $\hat{u}$ and $\hat{w}$ are dependent not only on $\hat{v}$ but also on the vertical vorticity $\hat{\eta}$. However, because $\hat{\eta}$ decays rapidly outside the boundary layer, in this region $\hat{u}$ and $\hat{w}$ are basically a function of $\hat{v}$. It was suggested that the influence of $\hat{\eta}$ on the oblique waves might be connected with the stronger activity observed inside the boundary layer.

Motivated by the results of Cohen *et al.*, the waves satisfying Craik's criterion were calculated for the experimental results presented in §2. The dominant two-dimensional Tollmien–Schlichting wave was considered to have a non-dimensional frequency $\beta = 0.1$ and the resonant oblique waves were obtained by solving equation (7) together with the resonant conditions (16).

To map out the dispersion relation it is usually necessary to solve the Orr–Sommerfeld equation a large number of times. Here a faster way of calculating eigenvalues from a double complex summation developed by Gaster (1978*b*) was used. The procedure involved only the inversion of the summation, which was done numerically through a complex Newton–Raphson root-finding algorithm. This summation was developed for two-dimensional modes. The eigenvalues of three-dimensional modes were found in terms of two-dimensional ones by using the relations

$$\tilde{\alpha}_x^2 = \alpha_x^2 + \alpha_z^2, \tag{23}$$

$$\tilde{\beta} = \frac{\beta \tilde{\alpha}_x}{\alpha_x}, \tag{24}$$

$$\tilde{R} = \frac{R \alpha_x}{\tilde{\alpha}_x}, \tag{25}$$

where the tildes indicate the two-dimensional modes. These relations were first given by Squire (1933) for temporal modes and can be readily obtained from equation (4). For spatial modes the same relations apply, but in this case $\tilde{R}$ has to be complex.

The results are shown by the asterisks in figure 4. It appeared that, at least for some packets, the spanwise wavenumbers of the waves satisfying the criterion for three-wave resonance closely coincided with the oblique waves observed in the experiments. However, the dominant frequency of the oblique waves was not quite half that of the fundamental, but slightly higher. The secondary instability mechanism can also amplify waves that do not have exactly the subharmonic frequency, in which case these waves are called detuned modes. However, the detuned modes must occur in pairs symmetric with respect to the subharmonic frequency, that is one wave with frequency $\frac{1}{2}f + \Delta f$ and another with $\frac{1}{2}f - \Delta f$ where f is the fundamental frequency (Kachanov & Levchenko 1984). This did not seem to be the case in the current experiments.

The apparent disagreement with the theory might be attributed to the fact that in the boundary layer the Reynolds number changes downstream with the evolution of the packet. In fact, as the wavepackets travel the fundamental dominant modes shift towards lower frequencies. When the resonance is triggered the oblique waves start to amplify, but by the time they are large enough to be measured the fundamental modes that triggered the process might have decayed considerably. However, this argument is questionable because the non-dimensional frequencies of the wavepackets did not change significantly, figure 3, suggesting that there is in fact a real mismatch between the frequency bands.

One feature of the experimental results still needs to be examined, namely the dependence of the nonlinear evolution of the wavepackets on the phase composition. The theories presented did not include phase effects, but extension of the theories (Monkewitz 1988; Healey 1995) and experiments on plane wavetrains (Hajj, Miksad & Powers 1993; Kachanov 1994) have shown that the resonant mechanisms are dependent on the relative phase of the waves involved. For instance, the experiments mentioned have demonstrated that when a two-dimensional wave is artificially excited and the subharmonic waves arise from the background noise, the resonant interaction does not occur uniformly, but displays an intermittent character. The reason for this may not arise solely through the amplitude modulation, but also from the irregular phase composition of the background noise. When the subharmonic modes are artificially excited together with the fundamental ones, variation of the phase relation advances or delays the onset of the resonant interaction.

Measurements from wavepackets with different phases appeared to support this scenario. Moreover, the pattern of dependence of the nonlinear evolution on the phase composition of the packet, as well as the repeatability of the results, indicate that, if the process is one of resonance, the subharmonic modes did not arise from the random background noise. Instead, they must have come from a deterministic source. It appeared that the subharmonic seeds for the resonant interaction were the low-frequency modes in the linear wavepacket, which were excited by the point source.

Despite the reasonably good agreement between the prediction from the resonant mechanism and the experiments, it is in fact difficult to draw a definitive conclusion as to whether the process is one of resonance. One of the reasons is that, possibly because the wavepacket is composed of a relatively broad band of fundamental frequencies, the oblique waves that appear in the experiments cover a wide range of spanwise wavenumbers. In view of the uncertainties involved, it is not significant that the theoretical prediction fall close to this band. The fact that, in the evolution of the wavepackets, the dominant fundamental frequencies change with downstream distance contributes to some further blurring of the picture.

5. Further experiments

It appeared that more conclusive results were needed to clarify the issue, and therefore an experiment was set up to investigate whether subharmonic resonance could provide a complete explanation of the observations. Time series for exciting the flow similar to those used in Medeiros & Gaster (1999) were constructed, but with the Fourier components associated with the subharmonic modes removed. Hereafter these time series will be referred to as truncated excitation time series as opposed to the previously used ones, which will be called complete excitation time series. Experiments were performed to compare the evolution of wavepackets generated with the two different types of time series. Both resonant mechanisms discussed are dependent on the initial amplitudes of the subharmonic modes. In Craik's mechanism this dependence is expressed by equations (17) to (19). The secondary instability is governed by a homogeneous equation with homogeneous boundary conditions and therefore requires a non-zero initial condition for a non-trivial solution. The outcome of spatial amplification of an initial secondary instability is crucially dependent on the amplitude of the seed. If the subharmonic modes were seeded by the artificial excitation in some way, removing the subharmonic seed should affect the nonlinear evolution. Moreover, if resonance were observed in the experiments with the truncated excitation time series, the subharmonic mode would have had to arise from the background noise. Therefore the dependence of the nonlinear evolution on the phase would not follow a pattern. In fact, in this case the results would not be deterministic.

The experiments presented in the previous section indicated that the fundamental and the subharmonic spectral band are not entirely separated, but, instead, displayed an overlap region. The problem is further complicated by the fact that the range of frequencies composing the bands varied with Reynolds number. Therefore, selection of the Fourier components to be removed from the excitation time series was a matter of some debate. The criterion adopted here was to remove a band that contained very little energy but which still included the dominant modes of the subharmonic band. The evolution of the positive packet generated from the two different time series were compared in the Fourier domain, figure 5. First, it was observed that the truncated time series did not introduce energy in the low-frequency band, whereas the complete time series did. This is clearly indicated by the spectra at station $x = 0.300\,\mathrm{m}$. The frequency is a parameter that is conserved with streamwise propagation. Therefore, the verification that the truncated time series did not excite the subharmonic modes at the centreline ensures that these modes were not excited by the point source throughout the packet. The evolution in the physical domain is shown in figure 6, while figure 7 shows in Fourier space a comparison for packets of different phases.

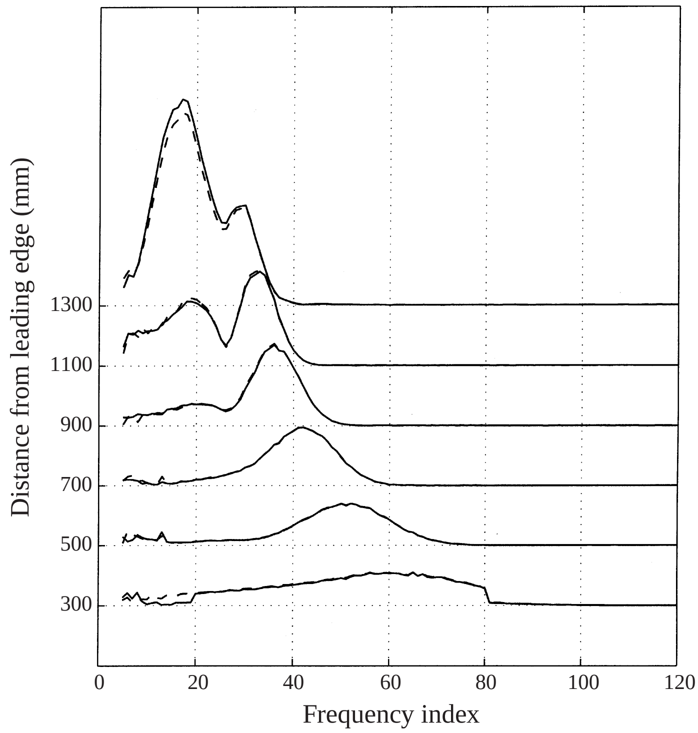

FIGURE 5. Fourier domain. Comparison of the evolution of a wavepacket created with the complete excitation time series (dashed lines) and a wavepacket created with the truncated excitation time series (solid lines).

In figure 5 it is found that at $x = 0.5$ m both the truncated and complete excitation response have virtually identical output. This is also shown by the fact that the subsequent nonlinear evolution is insensitive to whether the excitation was truncated or not. This occurs for all the packets of different phase, figure 7. What happens between $x = 0.3$ and 0.5 m cannot be explained by a subharmonic mechanism because this is sensitive to the initial amplitude, that is the seed, of the subharmonic. Moreover, the subharmonic mechanism is sensitive to phase. If it were responsible for the output at 0.5 m it would not have produced identical outputs because the seed of the truncated series would have been non-deterministic. Therefore it is concluded that some mechanism of production of low-frequency waves occurs between $x = 0.3$ and 0.5 m that generates the deterministic signal at $x = 0.5$ m. It is possible that for the subsequent nonlinear evolution, say, from $x = 0.5$ m onwards, the mechanism is of the subharmonic resonance type, but it is not at all clear how the low-frequency band is generated between $x = 0.3$ m and 0.5 m.

In summary, the nonlinear regime of both wavepackets and wavetrains features the appearance of oblique modes with frequency lower than the Tollmien–Schlichting band. In the case of two-dimensional regular wavetrains, theories have linked this phenomenon with the amplification of subharmonic modes that are already present in the flow, either in the background noise or due to artificial excitation. The experiments discussed in this paper demonstrated that for the wavepackets the oblique low-frequency waves arose neither from the background nor from the artificial excitation. It was therefore concluded that the process involved the production of oblique modes.

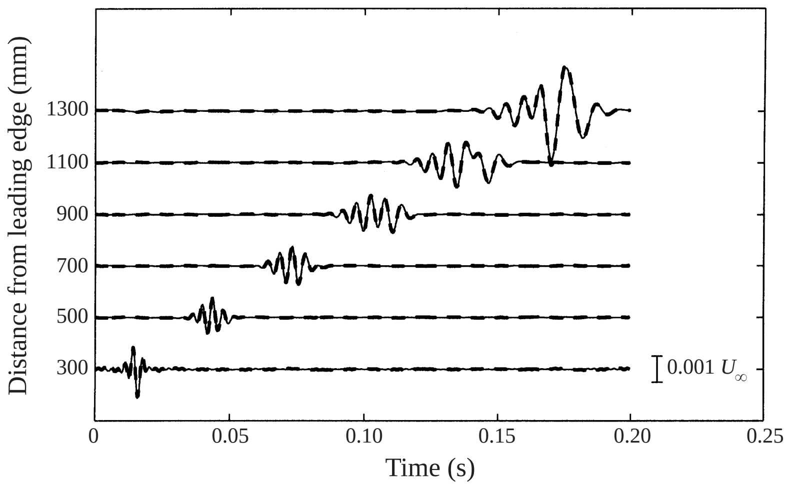

FIGURE 6. Physical domain. Comparison of the evolution of a wavepacket created with the complete excitation time series (thick dashed lines) and a wavepacket created with the truncated excitation time series (thin lines).

6. Conclusion and final remarks

Previous studies (Cohen *et al.* 1991; Konzelmann 1990) have linked the nonlinear regime of the wavepackets to some type of subharmonic resonance. In the present work we extend these studies by investigating more fully whether these mechanisms could explain the experimental observations, in particular the strong influence of the phase on the nonlinear evolution of the packets.

The current experiment on full three-dimensional packets confirmed the observation made in Medeiros & Gaster (1999) that the evolution is strongly sensitive to the phase of the ripples within the packets. For all the packets however, the nonlinear regime was characterized by the appearance of low-frequency oblique modes of similar streamwise and spanwise wavenumbers. From these results it is unclear whether these modes are subharmonics of the fundamental waves. The system is composed of a large number of waves and the non-parallel effects further blur the picture.

The repeatability of the results for the different packets and the consistent pattern of dependence on the phase made it clear that the oblique modes could not have arisen from the random background noise present in the tunnel. It appeared that these oblique modes could only have been seeded by the deterministic, artificial excitations. New experiments were then specially designed to verify this possibility. Excitations were used that did not contain components in the range of frequency of the dominant oblique modes. The results showed, however, that the phenomenon was not affected by the low-frequency oblique modes excited at the source. Therefore the deterministic oblique modes must have come from some as yet unidentified mechanism of wave production.

The production of modes is generally associated with harmonics of the fundamental or mean flow distortion, which arise from self-interaction of modes via the Reynolds stress terms. But it is important to note that in a system composed of a large number of modes, like the wavepackets, the Reynolds stresses also cover a wide range of the spectrum and production of modes of virtually all other frequencies becomes possible.

Finally, it is important to emphasize that the results presented here did not rule out the possibility of subharmonic resonance in the system. They could only show that

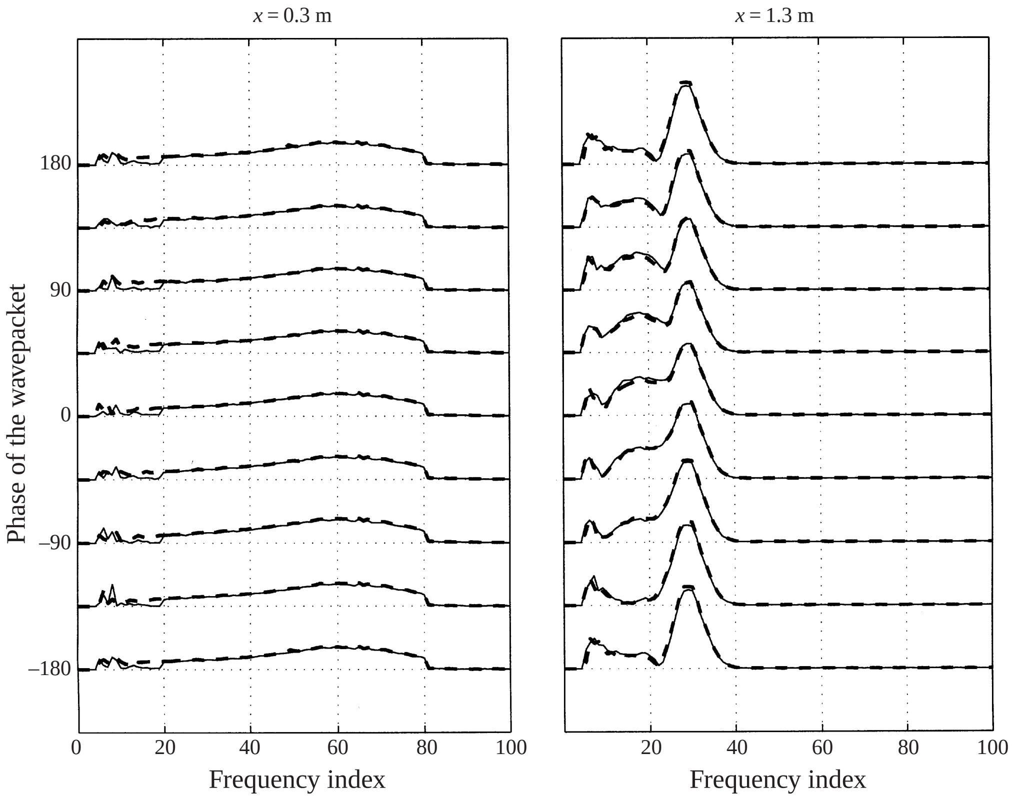

FIGURE 7. Fourier domain. Comparison of the nonlinear wavepackets created with the complete excitation time series (thick dashed lines) and the nonlinear wavepackets created with the truncated excitation time series (thin lines) for different wavepacket phases.

subharmonic resonance alone cannot explain the observations. In fact, it is possible that the nonlinearly produced waves in turn resonate with the fundamental modes. Since the resonant mechanisms are sensitive to the relative phase of the resonant modes, this might well explain the influence of the phase on the nonlinear behaviour of the packets. This conjecture is however as yet unproven.

The authors want to acknowledge the fruitful discussions with Dr F. N. Shaikh. This work was supported by a grant from CNPq/Brazil, No. 20012691-1 and by an ORS award from UK.

REFERENCES

BREUER, K. S., COHEN, J. & HARITONIDIS, J. H. 1997 The late stages of transition induced by a low-amplitude wavepacket in a laminar boundary layer. *J. Fluid Mech.* **340**, 395–411.

COHEN, J. 1994 The initial evolution of a wave packet in a boundary layer. *Phys. Fluids* **6**, 1133–1143.

COHEN, J., BREUER, K. S. & HARITONIDIS, J. H. 1991 On the evolution of a wave packet in a laminar boundary layer. *J. Fluid Mech.* **225**, 575–606.

CORKE, T. C. & MANGANO, R. A. 1989 Resonant growth of three-dimensional modes in transitioning Blasius boundary layers. *J. Fluid Mech.* **209**, 93–150.

CRAIK, A. D. D. 1971 Nonlinear resonant instability in a boundary layer. *J. Fluid Mech.* **50**, 393–413.

CRAIK, A. D. D. 1985 *Wave Interaction and Fluid Flows*. Cambridge University Press.

DRAZIN, P. G. & REID, W. H. 1981 *Hydrodynamic Stability*. Cambridge University Press.

GASTER, M. 1962 A note on the relation between temporally-increasing and spatially-increasing disturbances in hydrodynamic instability. *J. Fluid Mech.* **14**, 222–224.

GASTER, M. 1965 On the generation of spatially growing waves in a boundary layer. *J. Fluid Mech.* **22**, 433–441.

GASTER, M. 1975 A theoretical model of a wave packet in the boundary layer on a flat plate. *Proc. R. Soc. Lond.* A **347**, 271–289.

GASTER, M. 1978*a* The physical process causing breakdown to turbulence. In *12th Naval Hydrodynamics Symp., Washington.*

GASTER, M. 1978*b* Series representation of the eigenvalues of the Orr–Sommerfeld equation. *J. Comput. Phys.* **29**, 147–162.

GASTER, M. & GRANT, I. 1975 An experimental investigation of the formation and development of a wavepacket in a laminar boundary layer. *Proc. R. Soc. Lond.* A **347**, 253–269.

GRIMSHAW, R. 1990 *Nonlinear Ordinary Differential Equations.* Blackwell Scientific Publications.

HAJJ, M. R., MIKSAD, R. W. & POWERS, E. J. 1993 Fundamental-subharmonic interaction: effect of phase relation. *J. Fluid Mech.* **256**, 403–426.

HEALEY, J. J. 1995 A new boundary resonance enhanced by wave modulation: theory and experiment. *J. Fluid Mech.* **304**, 231–262.

HERBERT, T. 1983 Secondary instability of plane channel flow to subharmonic three-dimensional disturbances. *Phys. Fluids* **26**, 871–874.

HERBERT, T. 1988 Secondary instability of boundary layers. *Ann. Rev. Fluid Mech.* **20**, 487–526.

KACHANOV, Y. S. 1994 Physical mechanisms of laminar boundary layer transition. *Ann. Rev. Fluid Mech.* **26**, 411–482.

KACHANOV, Y. S. & LEVCHENKO, V. Y. 1984 The resonant interaction of disturbances at laminar-turbulent transition in a boundary layer. *J. Fluid Mech.* **138**, 209–247.

KONZELMANN, U. 1990 Numerische Untersuchungen zur räumlichen Entwicklung dreidimensionaler Wellenpakete in einer Plattengrenzschichtströmung. PhD thesis, Universität Stuttgart.

KONZELMANN, U. & FASEL, H. 1991 Numerical simulation of a three-dimensional wave packet in a growing flat plate boundary layer. In *Boundary layer Transition and Control*, pp. 24.1–24.11. The Royal Aeronautical Society, Cambridge.

MACK, L. M. 1984 Boundary-layer linear stability theory, *Special course on Stability and Transition of Laminar Flow, AGARD Rep.* 709.

MEDEIROS, M. A. F. 1996 The nonlinear behaviour of modulated Tollmien–Schlichting waves. PhD thesis, Cambridge University.

MEDEIROS, M. A. F. & GASTER, M. 1999 The influence of phase on the nonlinear evolution of wavepackets in boundary layers, *J. Fluid Mech.* **397**, 259–283.

MONKEWITZ, P. 1988 Subharmonic resonance, pairing and shredding in the mixing layer. *J. Fluid Mech.* **188**, 223–252.

SHAIKH, F. N. 1993 Turbulent spot in a transitional boundary layer. PhD thesis, Cambridge University.

SQUIRE, H. B. 1933 On the stability of three-dimensional distribution of viscous fluid between parallel walls. *Proc. R. Soc. Lond.* A **142**, 621–628.